An inhomogeneous, L^2 critical, nonlinear Schrödinger equation

François Genoud
Department of Mathematics and
The Maxwell Institute for Mathematical Sciences
Heriot-Watt University
Edinburgh, EH14 4AS, Scotland
F.Genoud@hw.ac.uk

October 6, 2011

Abstract

An inhomogeneous nonlinear Schrödinger equation is considered, that is invariant under L^2 scaling. The sharp condition for global existence of H^1 solutions is established, involving the L^2 norm of the ground state of the stationary equation. Strong instability of standing waves is proved by constructing self-similar solutions blowing up in finite time.

This paper will appear in the Zeitschrift für Analysis und ihre Anwendungen.

1 Introduction

The purpose of this note is to point out the case of an inhomogeneous nonlinear Schrödinger equation having L^2 scaling invariance. Namely, we consider the Cauchy problem

$$i\partial_t\phi + \Delta\phi + |x|^{-b}|\phi|^{2\sigma}\phi = 0, \quad \phi(0,\cdot) = \phi_0 \in H^1(\mathbb{R}^N) \tag{NLS}$$

with $\sigma = (2-b)/N$, in any dimension $N \geq 1$. Here and henceforth, $H^1(\mathbb{R}^N)$ denotes the Sobolev space of complex-valued functions $H^1(\mathbb{R}^N, \mathbb{C})$, with its usual norm. We suppose that $0 < b < \min\{2, N\}$. The case $b = 0$ is the classical (focusing) nonlinear Schrödinger equation with L^2 critical nonlinearity. In the above setting, it turns out that (NLS) is also invariant under the L^2 scaling

$$\phi \to \phi_\lambda(t,x) := \lambda^{N/2}\phi(\lambda^2 t, \lambda x), \quad \phi_0 \to (\phi_0)_\lambda(x) := \lambda^{N/2}\phi_0(\lambda x) \quad \text{for } \lambda > 0. \tag{1.1}$$

We came across this (modified) critical nonlinearity for (NLS) while studying stability of standing waves for some classes of nonlinear Schrödinger equations, where (NLS) arises both as a model and a limiting case, see [7, 4, 5]. In particular, the Cauchy problem (NLS) is studied there, and it is found that, for $0 < b < \min\{2, N\}$, it is well-posed in $H^1(\mathbb{R}^N)$,

locally if $0 < \sigma < \widetilde{2} := \begin{cases} (2-b)/(N-2) & \text{if } N \geq 3, \\ \infty & \text{if } N \in \{1, 2\}; \end{cases}$

globally for small initial conditions if $0 < \sigma < \widetilde{2}$;

globally for any initial condition in $H^1(\mathbb{R}^N)$ if $0 < \sigma < \dfrac{2-b}{N}$.

Theorem 1 below answers the natural question: in the limit case $\sigma = (2-b)/N$, how small should the initial condition be to have global existence? We consider here strong solutions $\phi = \phi(t,x) \in C_t^0 H_x^1([0,T) \times \mathbb{R}^N)$ for some $T > 0$, and the notion of well-posedness as defined in [1]. Our notation for the space-time function spaces comes from [13]. We may simply denote by $\phi(t) \in H^1(\mathbb{R}^N)$ the function $x \to \phi(t,x)$. The solution is called global (in time) if we can take $T = \infty$. If it is not the case, the blowup alternative states that $\|\phi(t)\|_{H^1} \to \infty$ as $t \uparrow T$. Moreover, we have conservation of the L^2 norm along the flow of (NLS),

$$\|\phi(t)\|_{L_x^2} = \|\phi_0\|_{L_x^2} \quad \text{for all } t \in [0,T),$$

and of the energy

$$E(\phi(t)) := \int_{\mathbb{R}^N} |\nabla\phi(t)|^2 \, \mathrm{d}x - \frac{1}{\sigma+1} \int_{\mathbb{R}^N} |x|^{-b} |\phi(t)|^{2\sigma+2} \, \mathrm{d}x = E(\phi_0) \quad \text{for all } t \in [0,T). \tag{1.2}$$

Also, the L^2 norm of $\phi(t)$ is invariant under the transformation (1.1), $\|\phi(t)\|_{L_x^2} = \|\phi_\lambda(t)\|_{L_x^2}$. This is why it is called the L^2 scaling.

A standing wave for (NLS) is a (global) solution of the form $\varphi_\omega(t,x) = \mathrm{e}^{i\omega^2 t} u_\omega(x)$ for some $\omega \in \mathbb{R}$, with $u_\omega \in H^1(\mathbb{R}^N)$ satisfying the stationary equation

$$\Delta u - \omega^2 u + |x|^{-b} |u|^{2\sigma} u = 0. \tag{E_ω}$$

In [7, 4, 5], we were concerned with bifurcation and orbital stability of standing waves for nonlinear Schrödinger equations with inhomogeneous nonlinearities of the form $V(x)|\phi|^{2\sigma}\phi$ with $V(x) \sim |x|^{-b}$ at infinity or around the origin. These equations have important applications in nonlinear optics (see [5]). The limiting problem (NLS) turned out to play a central role in our analysis. For this model case, a global branch of positive solutions of (E_ω) is simply given by the mapping $u \in C^1((0,\infty), H^1(\mathbb{R}^N))$,

$$\omega \mapsto u_\omega(x) = u(\omega)(x) := \omega^{\frac{2-b}{2\sigma}} u_1(\omega x), \tag{1.3}$$

where u_1 is the unique positive radial solution (ground state) of (E_ω) with $\omega = 1$. The existence of the ground state is proved in [7, 4] by variational methods in dimension $N \geq 2$, and in [5] for $N = 1$. Uniqueness is a delicate problem, handled in dimension $N \geq 3$ by a theorem of Yanagida [17] (see [4]), in dimension $N = 2$ by a shooting argument [6], and in dimension $N = 1$ by the method of horizontal separation of graphs of Peletier and Serrin [11], as used in [14]. These existence and uniqueness results hold for $0 < b < \min\{2, N\}$ and $0 < \sigma < \widetilde{2}$.

Using the general theory of orbital stability of Grillakis, Shatah and Strauss [9], we obtained in [7, 4, 5] various stability/instability results for general nonlinearities $V(x)|\phi|^{2\sigma}\phi$ by studying the monotonicity of the L^2 norm of the standing waves, as a function of $\omega > 0$. It turned out that $\sigma = (2-b)/N$ is a threshold for stability in the regimes we considered. For this value of σ, we could not determine if the standing waves are stable or not, even in the model case $V(x) = |x|^{-b}$. In fact, if $\sigma = (2-b)/N$, we have $\|u_\omega\|_{L^2} = \|u_1\|_{L^2}$ along the curve of solutions (1.3), for u_ω is then an L^2 scaling of u_1. In Section 3, we prove a strong instability result for standing waves of (NLS), without requiring that u_ω be the ground state of (E_ω).

Section 2 is devoted to a sharp global existence result in the spirit of Weinstein [15]. For $\sigma = (2-b)/N$ we prove that the solutions of (NLS) are global in time provided $\|\phi_0\|_{L^2} < \|\psi\|_{L^2}$,

where ψ is the ground state of (E_1). This is done by computing the best constant for an interpolation inequality. The sharpness of the result is proved in Section 3 where we construct self-similar solutions blowing up in finite time, in particular with the critical mass $\|\psi\|_{L^2}$.

Related results for inhomogeneous nonlinear Schrödinger equations can be found in the literature, see for instance [10] and [3]. However, no one seems to have noticed the possibility of L^2 scaling invariance. The results established here use basic ideas going back to [15, 16]. The classical L^2 critical case ($b = 0$) has been studied extensively, and in particular the properties of the blowup solutions are quite well-known (see [12] for a survey). The case $b \neq 0$ certainly deserves further investigation.

Notation. In Section 2 we work in the Sobolev space of real-valued functions $H := H^1(\mathbb{R}^N, \mathbb{R})$. We use the shorthand notation $\|\cdot\|_p := \|\cdot\|_{L^p}$ for the usual Lebesgue norms throughout.

2 Critical mass and global existence

We start by solving the minimization problem

$$\inf_{u \in H \setminus \{0\}} J(u) \tag{2.1}$$

where $J : H \setminus \{0\} \to \mathbb{R}$ is the Weinstein functional defined by

$$J(u) = J_{N,b}(u) = \frac{\|\nabla u\|_2^2 \|u\|_2^{2\sigma}}{I(u)} \quad \text{for } \sigma = \frac{2-b}{N}, \tag{2.2}$$

with

$$I(u) = \int_{\mathbb{R}^N} |x|^{-b} |u|^{2\sigma+2} \, \mathrm{d}x. \tag{2.3}$$

Lemma 1 *For $N \geq 1$, $0 < b < \min\{2, N\}$ and $0 < \sigma < \widetilde{2}$, the functional $I : H \to \mathbb{R}$ defined in* (2.3) *is of class $C^1(H, \mathbb{R})$ and is weakly sequentially continuous. In particular, it follows that $J \in C^1(H \setminus \{0\}, \mathbb{R})$.*

Proof. See [7, Section 2.1] and [4, Section 1.1] for $N \geq 2$, [5, Section 2] for $N = 1$. □

Proposition 1 *Let $N \geq 1$, $0 < b < \min\{2, N\}$ and $\sigma = (2 - b)/N$. There exists a positive radial function $\psi \in H$ such that:*

(i) ψ is a minimizer for (2.1)*, that is, $J_{N,b}(\psi) = \inf_{u \in H \setminus \{0\}} J_{N,b}(u)$;*

(ii) ψ is the unique ground state of ($\mathrm{E}_{\sqrt{\sigma}}$).

Furthermore, the minimum value is $J_{N,b}(\psi) = \dfrac{\|\psi\|_2^{2\sigma}}{\sigma + 1} = \dfrac{\|\psi\|_2^{\frac{4-2b}{N}}}{\frac{2-b}{N} + 1}$.

Proof. We follow Weinstein [15]. Let $\{u_n\} \subset H \setminus \{0\}$ be a minimizing sequence for (2.1):

$$J(u_n) \to m := \inf J \geq 0 \quad \text{as } n \to \infty.$$

Clearly, we can choose $u_n \geq 0$. Moreover, by Schwarz symmetrization (see [7, p.146]) we can suppose that u_n is radial and radially non-increasing for all n. It follows from the structure of

$J = J_{N,b}$ that J is invariant under the scaling $u \to u_{\lambda,\mu}(x) := \lambda u(\mu x)$, $\lambda, \mu > 0$. (This is not the case for $\sigma \neq (2-b)/N$.) This allows us to choose u_n such that

$$\|\nabla u_n\|_2 = \|u_n\|_2 = 1 \quad \text{for all } n.$$

Hence there exists $u^* \in H$ such that, up to a subsequence, $u_n \rightharpoonup u^*$ weakly in H. Furthermore, u^* is non-negative, spherically symmetric, radially non-increasing, and

$$\|\nabla u^*\|_2 \leq 1 \quad \text{and} \quad \|u^*\|_2 \leq 1. \tag{2.4}$$

Now by Lemma 1 and (2.4) we have

$$m = \lim J(u_n) = \lim \frac{1}{I(u_n)} = \frac{1}{I(u^*)} \geq J(u^*) \tag{2.5}$$

so that, in fact, $J(u^*) = m$ and $\|\nabla u^*\|_2 = \|u^*\|_2 = 1$. In particular, $u_n \to u^*$ strongly in H. (Note that (2.5) prevents $u^* = 0$.) This concludes the proof of (i).

To show that ψ can be chosen so as to satisfy $(\mathrm{E}_{\sqrt{\sigma}})$, we first remark that u^* is a solution of the Euler-Lagrange equation corresponding to (2.1), which reads

$$\Delta u^* - \sigma u^* + m(\sigma+1)|x|^{-b}(u^*)^{2\sigma+1} = 0.$$

Setting $u^* = [m(\sigma+1)]^{-1/2\sigma}\psi$, it follows that ψ is a solution of $(\mathrm{E}_{\sqrt{\sigma}})$. Furthermore, ψ is positive and radial, so it is the unique ground state of $(\mathrm{E}_{\sqrt{\sigma}})$. $\square$

As an immediate consequence we have

Corollary 1 $C_{N,b} := \dfrac{\frac{2-b}{N}+1}{\|\psi\|_2^{\frac{4-2b}{N}}}$ *is the best constant for the inequality*

$$\int_{\mathbb{R}^N} |x|^{-b}|u|^{\frac{4-2b}{N}+2}\,\mathrm{d}x \leq C\|\nabla u\|_2^2\|u\|_2^{\frac{4-2b}{N}}, \quad u \in H. \tag{2.6}$$

Remark 1 Note that (2.6) is a special case of the interpolation inequalities obtained in [2].

We now turn to the global existence result.

Theorem 1 *Set* $\sigma = (2-b)/N$ *and let* ψ *be the ground state of* (E_1). *If*

$$\|\phi_0\|_2 < \|\psi\|_2,$$

the solution of (NLS) *is global and bounded in* H^1.

Proof. Local existence of solutions to (NLS) is ensured by results in [1] (see [7, Appendix K] for precise statements and references). So the maximal solution $\phi(t,x)$ of (NLS) with initial condition ϕ_0 is defined on a time interval $[0,T)$ with $T \in (0,\infty]$. Moreover, we have the conservation laws

$$E(\phi(t)) = E(\phi_0) \quad \text{and} \quad \|\phi(t)\|_2 = \|\phi_0\|_2 \quad \text{for all } t \in [0,T),$$

where E is defined in (1.2). It is well-known since [8] that the boundedness of $\|\nabla\phi(t)\|_2$ is then sufficient to conclude global existence. Using the constants of motion, we have

$$\begin{aligned}\|\nabla\phi(t)\|_2^2 &= E(\phi(t)) + \frac{1}{\sigma+1}\int_{\mathbb{R}^N} |x|^{-b}|\phi(t)|^{2\sigma+2}\,\mathrm{d}x \\ &\leq E(\phi_0) + \frac{C}{\sigma+1}\|\nabla\phi(t)\|_2^2\|\phi_0\|_2^{2\sigma},\end{aligned}$$

where $C = C_{N,b} > 0$ is the constant given by Corollary 1. Hence,

$$\left(1 - \frac{C_{N,b}}{\frac{2-b}{N}+1}\|\phi_0\|_2^{\frac{4-2b}{N}}\right)\|\nabla\phi(t)\|_2^2 \leq E(\phi_0). \tag{2.7}$$

Using the formula for $C_{N,b}$, it follows from (2.7) that the solution is global if $\|\phi_0\|_2 < \|\psi\|_2$ where ψ is the ground state of $(\mathrm{E}_{\sqrt{\sigma}})$. But for $\sigma = (2-b)/N$, $(\mathrm{E}_{\sqrt{\sigma}})$ is transformed into (E_1) by the scaling

$$\psi \to \psi_{\lambda^{-1}}(x) = \lambda^{-N/2}\psi(\lambda^{-1}x) \quad \text{with } \lambda = \sqrt{\sigma}.$$

Since this transformation leaves the L^2 norm unchanged, we can indeed choose ψ to be the ground state of (E_1). The proof is complete. □

Remark 2 We call $\|\psi\|_2$ the *critical mass* for (NLS). As we show below, the condition for global existence given by Theorem 1 is sharp in the sense that we can find solutions with critical mass which blow up in finite time.

3 Instability of standing waves

It is a lenghty but straightforward calculation to show that (NLS) is invariant under the pseudoconformal transformation, as defined in [1, Section 6.7]. Namely, for any $a \in \mathbb{R}$, if $\phi(s,y) \in C_s^0 H_y^1([0,S)\times\mathbb{R}^N)$ is a solution to (NLS) (with the obvious modification of the variables), then the function $\phi_a(t,x) \in C_t^0 H_x^1([0,T)\times\mathbb{R}^N)$ defined by

$$\phi_a(t,x) = (1-at)^{-\frac{N}{2}}\mathrm{e}^{-i\frac{a|x|^2}{4(1-at)}}\phi\left(\frac{t}{1-at},\frac{x}{1-at}\right) \quad \text{with} \quad T = \begin{cases}\infty & \text{if } aS \leq -1 \\ \frac{S}{1+aS} & \text{if } aS > -1\end{cases} \tag{3.1}$$

is also a solution. The fact that (NLS) with $\sigma = (2-b)/N$ behaves nicely under (3.1) when $b > 0$ is closely related to the L^2 scaling invariance of the equation. In fact, the pseudoconformal transformation conserves the L^2 norm:

$$\|\phi_a(t)\|_2 \equiv \|\phi(s)\|_2.$$

Using (3.1), we now show that all standing waves for (NLS) with $\sigma = (2-b)/N$ are strongly unstable in the following sense. By scaling, it is enough to consider the case $\omega = 1$.

Theorem 2 *Let $u \in H$ be a nontrivial solution of* (E_1). *For any $\delta > 0$ there exists a solution $\varphi \in C_t^0 H_x^1([0,T)\times\mathbb{R}^N)$ of* (NLS) *such that $\|\varphi(0) - u\|_{H^1} < \delta$ and $\|\varphi(t)\|_{H^1} \to \infty$ as $t \uparrow T$.*

Proof. Let $a > 0$ to be tuned later. We apply the transformation (3.1) to the standing wave $\phi(t,x) = \mathrm{e}^{it}u(x)$, defining $\varphi \in C_t^0 H_x^1([0,a^{-1}) \times \mathbb{R}^N)$ by ($S = \infty$ for ϕ):

$$\varphi(t,x) = (1-at)^{-\frac{N}{2}} \mathrm{e}^{-i\frac{a|x|^2}{4(1-at)}} \mathrm{e}^{i\frac{t}{1-at}} u\left(\frac{x}{1-at}\right). \tag{3.2}$$

It is easy to check that

$$(1-at)\|\nabla\varphi(t)\|_2 \to \|\nabla u\|_2 \quad \text{as } t \uparrow a^{-1}$$

and so φ blows up at finite time $T := a^{-1}$. Furthermore, $\varphi(0,x) = \mathrm{e}^{-i\frac{a|x|^2}{4}}u(x)$ and we have:

$$\|\varphi(0) - u\|_2^2 = \int_{\mathbb{R}^N} |\mathrm{e}^{-i\frac{a|x|^2}{4}} - 1|^2 u(x)^2 \,\mathrm{d}x \tag{3.3}$$

$$\text{and} \quad \|\nabla\varphi(0) - \nabla u\|_2^2 = \int_{\mathbb{R}^N} |\mathrm{e}^{-i\frac{a|x|^2}{4}} - 1|^2 |\nabla u(x)|^2 + \frac{a^2}{4}|x|^2 u(x)^2 \,\mathrm{d}x. \tag{3.4}$$

It is standard to show that u decays exponentially and it follows by dominated convergence that both (3.3) and (3.4) go to zero as $a \to 0$. Hence, for any $\delta > 0$, there is $a_\delta > 0$ such that $\|\varphi(0) - u\|_{H^1} < \delta$ whenever $0 < a < a_\delta$. This concludes the proof. $\square$

Remark 3

(i) We know precisely the blowup rate of φ,

$$\|\varphi(t)\|_{H^1} \sim (1-at)^{-1} \quad \text{and} \quad \|\varphi(t)\|_\infty \sim (1-at)^{-N/2} \quad \text{as } t \uparrow a^{-1}.$$

(ii) The type of solutions constructed in (3.2) are often called 'self-similar' in the literature. In fact, the modulus $|\varphi(t,x)| = (1-at)^{-N/2}|u(x/(1-at))|$ presents a self-similar profile in the usual sense: at any time t, there is a scaling parameter $\lambda(t) > 0$ such that $|u(x)| = \lambda(t)^{N/2}|\varphi(t,\lambda(t)x)|$. Thus $|\varphi(t)|$ retains the shape of $|u|$ while blowing up.

Corollary 2 *There exists a solution of* (NLS) *with critical mass that blows up in finite time.*

Proof. Take φ defined by (3.2) with $u = \psi$, the ground state of (E_1). $\square$

Remark 4 Note that (3.2) yields blowup solutions with self-similar profiles corresponding to any solution of (E_1). In particular, it follows by Theorem 1 that ψ is the solution of (E_1) with minimal L^2 norm, as is well-known in the case $b = 0$.

References

[1] T. Cazenave, *Semilinear Schrödinger Equations.* Courant Lecture Notes in Mathematics, 10. American Mathematical Society, Providence, RI, 2003.

[2] L. Caffarelli, R. Kohn and L. Nirenberg, First order interpolation inequalities with weights. *Compositio Math.* **53** (1984), 259–275.

[3] J. Chen and B. Guo, Sharp global existence and blowing up results for inhomogeneous Schrödinger equations. *Discrete Contin. Dyn. Syst. Ser. B* **8** (2007), 357–367.

[4] F. Genoud, *Théorie de bifurcation et de stabilité pour une équation de Schrödinger avec une non-linéarité compacte.* PhD Thesis no. 4233, EPFL, 2008.

[5] F. Genoud, Bifurcation and stability of travelling waves in self-focusing planar waveguides. *Adv. Nonlinear Stud.* **10** (2010), 357–400.

[6] F. Genoud, A uniqueness result for $\Delta u - \lambda u + V(|x|)u^p = 0$ on $\mathbf{R}^2$. to appear in *Adv. Nonlinear Stud.*

[7] F. Genoud and C.A. Stuart, Schrödinger equations with a spatially decaying nonlinearity: existence and stability of standing waves. *Discrete Contin. Dyn. Syst.* **21** (2008), 137–186.

[8] J. Ginibre and G. Velo, On a class of nonlinear Schrödinger equations. I. The Cauchy problem, general case. *J. Funct. Anal.* **32** (1979), 1–32.

[9] M. Grillakis, J. Shatah and W. Strauss, Stability theory of solitary waves in the presence of symmetry I. *J. Funct. Anal.* **74** (1987), 160–197.

[10] F. Merle, Nonexistence of minimal blow-up solutions of equations $iu_t = -\Delta u - k(x)|u|^{4/N}u$ in $\mathbf{R}^N$. *Ann. Inst. H. Poincar Phys. Théor.* **64** (1996), 33–85.

[11] L.A. Peletier and J. Serrin, Uniqueness of positive solutions of semilinear equations in $\mathbf{R}^n$. *Arch. Rational Mech. Anal.* **81** (1983), 181–197.

[12] P. Raphaël, *Stability and Blow Up for the Nonlinear Schrödinger Equation.* Lecture notes for the Clay summer school on evolution equations, ETH, Zurich, 2008.

[13] T. Tao, *Nonlinear Dispersive Equations. Local and Global Analysis.* CBMS Regional Conference Series in Mathematics, 106. American Mathematical Society, Providence, RI, 2006.

[14] J.F. Toland, Uniqueness of positive solutions of some semilinear Sturm-Liouville problems on the half line. *Proc. Roy. Soc. Edinburgh Sect. A* **97** (1984), 259–263.

[15] M.I. Weinstein, Nonlinear Schrödinger equations and sharp interpolation estimates. *Comm. Math. Phys.* **87** (1982/83), 567–576.

[16] M.I. Weinstein, On the structure and formation of singularities in solutions to nonlinear dispersive evolution equations. *Comm. Partial Differential Equations* **11** (1986), 545–565.

[17] E. Yanagida, Uniqueness of positive radial solutions of $\Delta u + g(r)u + h(r)u^p = 0$ in $\mathbf{R}^n$. *Arch. Ration. Mech. Anal.* **115** (1991), 257–274.